

Similar Shapes

Introduction

- Dividing two lengths gives us the length scale factor
- Dividing two areas gives us the area factor
- Dividing two volumes gives us the volume factor

Notice, if we have area scale and want volume scale factor we need to find the length scale factor first. Similarly, if we have area scale and want volume scale factor and want the area scale factor we need to find the length scale factor first.

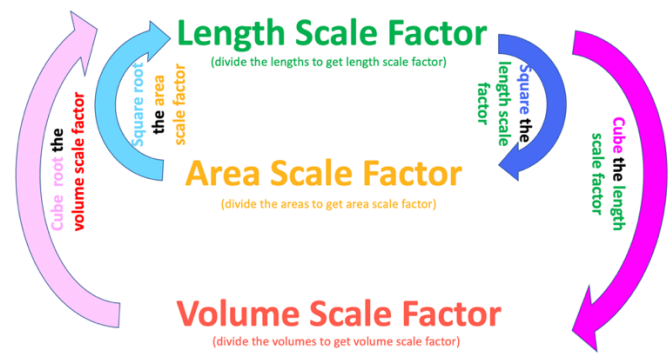
$$\text{Length scale factor} = \sqrt{\text{area scale factor}}$$

$$\text{Length scale factor} = \sqrt[3]{\text{volume scale factor}}$$

$$\text{Area Scale Factor} = (\text{Length Scale Factor})^2$$

$$\text{Volume Scale Factor} = (\text{Length Scale Factor})^3$$

In summary:

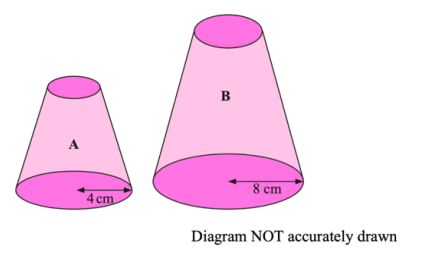


Now we can use the scale factors to multiply or divide and find whatever is missing. This will make sense once you see the examples below.

Easy Examples

Example 1

Two solid shapes, A and B, are mathematically similar. The base of shape A is a circle with radius 4 cm. The base of shape B is a circle with radius 8 cm. The surface area of shape A is 80 cm².



- Work out the surface area of shape B. The volume of shape B is 600 cm³
- Work out the volume of shape A

i. Here we have 2 lengths, so we can find the length scale factor first

Way 1: Divide the radius of B by the radius of A

$$\text{Length scale factor} = \frac{8}{4} = 2$$

$$\text{Area Scale Factor} = (\text{Length Scale Factor})^2 = 2^2 = 4$$

We have the surface area of A and want the surface area of B. B is bigger than A, hence we multiply A by 4

$$80 \times 4 = 320 \text{ cm}^2$$

Way 2: Divide the radius of A by the radius of B

$$\text{Length scale factor} = \frac{4}{8} = \frac{1}{2}$$

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We have the surface area of A and want the surface area of B. B is bigger than A, hence we divide A by $\frac{1}{4}$ to get a bigger number

$$80 \div \frac{1}{4} = 320 \text{ cm}^2$$

- Work out the surface area of shape B
- Work out the volume of shape A

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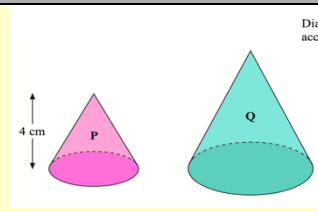
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Example 2

The two cones, P and Q, are mathematically similar. The total surface area of the cone P is 24 cm². The total surface area of the cone Q is 96 cm². The height of the cone P is 4 cm.



- Work out the height of the cone Q
- Work out the volume of cone Q

We have both areas so we can work out the area scale factor first. It doesn't matter which way around we divide to get the scale factor

Way 1: Divide area of Q by the area of P

$$\text{Area scale factor} = \frac{96}{24} = 4$$

$$\text{Length scale factor} = \sqrt{\text{area scale factor}} = \sqrt{4} = 2$$

We have the length of P and want the length of Q. Q is bigger than P, hence we multiply P by 2

$$4 \times 2 = 8 \text{ cm}$$

Way 2: Divide area of P by the area of Q

$$\text{Area scale factor} = \frac{24}{96} = \frac{1}{4}$$

$$\text{Length scale factor} = \sqrt{\text{area scale factor}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

We have the length of P and want the length of Q. Q is bigger than P, hence we divide P by $\frac{1}{2}$ in order to get a bigger number

$$4 \div \frac{1}{2} = 8 \text{ cm}$$

Way 1

$$\text{Volume Scale Factor} = (\text{Length Scale Factor})^3 = 2^3 = 8$$

We have the volume of P and want the volume of Q. Q is bigger than P, hence we multiply P by 8

$$12(8) = 96 \text{ cm}^3$$

Way 2

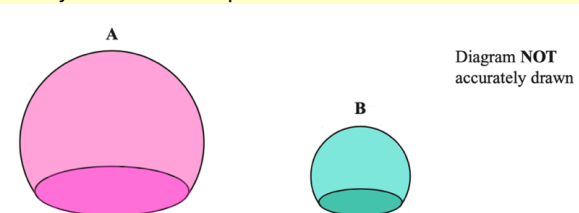
$$\text{Volume Scale Factor} = (\text{Length Scale Factor})^3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

We have the volume of P and want the volume of Q. Q is bigger than P, hence we divide P by $\frac{1}{8}$

$$12 \div \left(\frac{1}{8}\right) = 96 \text{ cm}^3$$

Example 3

A and B are two solid shapes which are mathematically similar. The shapes are made from the same material.



The volume of A is 540 grams. The volume of B is 20 grams. The surface area of A is 234 cm². Calculate the surface area of B.

We have the volumes of both shapes, so we can work out the volume scale factor. It doesn't matter which way we divide the areas to get this scale factor.

Way 1: Divide the volume of A by the volume of B

$$\text{Volume scale factor} = \frac{540}{20} = 27$$

We find Area Scale Factor by using Length Scale Factor

$$\text{Length scale factor} = \sqrt[3]{\text{Volume scale factor}} = \sqrt[3]{27} = 3$$

Hence, we can find

$$\text{Area Scale Factor} = (\text{Length Scale Factor})^2 = 3^2 = 9$$

We have the area of A and want the area of B. B is smaller than A, hence we divide A by 9 to get a smaller number

$$234 \div 9 = 26 \text{ cm}^2$$

Way 1: Divide the volume of A by the volume of B

$$\text{Volume scale factor} = \frac{20}{540} = \frac{1}{27}$$

We find Area Scale Factor by using Length Scale Factor

$$\text{Length scale factor} = \sqrt[3]{\text{Volume scale factor}} = \sqrt[3]{\frac{1}{27}} = \frac{1}{3}$$

Hence, we can find

$$\text{Area Scale Factor} = (\text{Length Scale Factor})^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$$

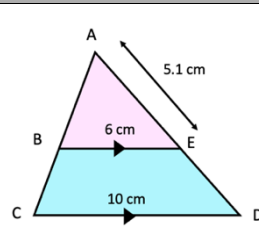
We have the area of A and want the area of B. B is smaller than A, hence we multiply A by $\frac{1}{9}$ to get a bigger number

$$234 \times \frac{1}{9} = 26 \text{ cm}^2$$

Medium Example

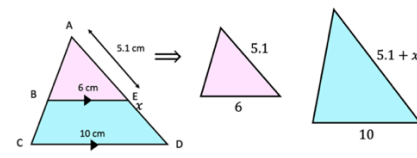
Example 1

ABC and AED are two straight lines. BE is parallel to CD.



- Calculate the length of DE
- Calculate the value of area of triangle ABE / area of trapezium BCDE

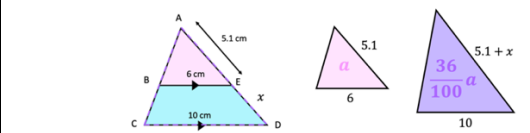
Hence we have



$$\frac{6}{10} = \frac{5.1}{5.1+x} \Rightarrow x = 3.4$$

$$\text{Length scale factor} = \frac{BE}{CD} = \frac{6}{10} = 0.6$$

$$\text{Area Scale Factor} = (\text{Length Scale Factor})^2 = (0.6)^2 = 0.36$$



$$\text{Blue area} = \frac{36}{100}a = \frac{64}{100}a$$

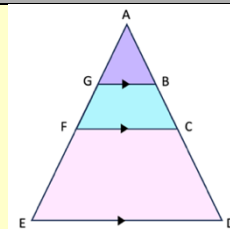
$$\text{Thus, we get } \frac{\text{Area of triangle ABE}}{\text{Area of trapezium BCDE}} = \frac{a}{64a} = \frac{25}{16}$$

Example 2

Here are three triangles, ABG, ACF and ADE. ABCD and AGFE are straight lines

$$AB : BC : CD = 1 : 2 : 3$$

Show that Area ABG : Area BCFG : Area CDEF = 1 : 8 : 27



Triangles AGB and AFC

Let's find length scale factor which allows us to find area scale factor after which is what we need

$$\text{Length scale factor} = \frac{AB}{AC} = \frac{1}{3}$$

$$\text{Area Scale Factor} = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$$

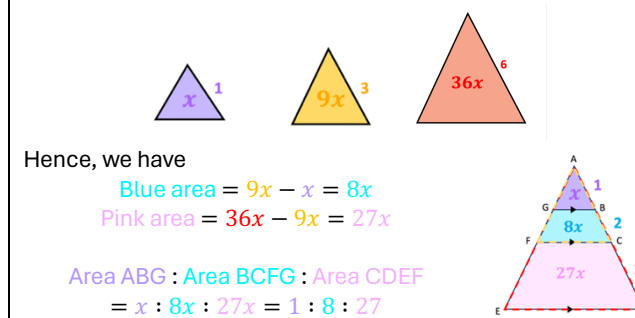
Triangles AGB and AED

Let's find length scale factor which allows us to find area scale factor after which is what we need

$$\text{Length scale factor} = \frac{AB}{AD} = \frac{1}{6}$$

$$\text{Area Scale Factor} = \left(\frac{1}{6}\right)^2 = \frac{1}{36}$$

We don't know the area of the purple, hence call it x. We can work out the areas in terms of x



Hence, we have

$$\text{Blue area} = 9x - x = 8x$$

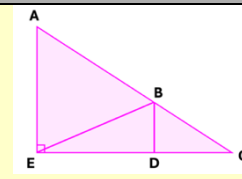
$$\text{Pink area} = 36x - 9x = 27x$$

$$\text{Area ABG} : \text{Area BCFG} : \text{Area CDEF} = x : 8x : 27x = 1 : 8 : 27$$

Somewhat Challenging

Example 1

AEC is a right-angled triangle. B is a point on AC and D is a point on EC. The area of triangle BCD is 5 cm². The area of triangle EBD is 15 cm². Work out the area of triangle ABE. You must show all your working.



First, notice that we know the area of these two triangles that share a common side BD. We can use this to our advantage.

$$\text{Area BDC} = 5 \quad \text{Area BDE} = 15$$

$$\frac{1}{2}(BD)(DC) = 5 \quad \frac{1}{2}(BD)(DE) = 15$$

$$\text{So, } BD = \frac{10}{DC} \quad \text{So, } BD = \frac{30}{DE}$$

$$\text{Hence, } BD \text{ is equal means } \frac{10}{DC} = \frac{30}{DE} \Rightarrow DE = 3DC$$

Let DC = x so DE = 3x. Finally, notice $\angle BCD = \angle ACE$ and the other pairs of angle are equal. Hence, AA criterion means $\triangle BCD \sim \triangle ACE$. Equal ratios of corresponding sides thus gives us

$$\frac{CD}{CE} = \frac{BD}{AE} = \frac{x}{x+3x} = \frac{y}{AE} \Rightarrow AE = 4y$$

Let's now use their areas to find the area scale factor too.

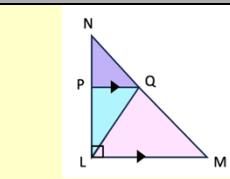
$$\text{Area Scale Factor} = \frac{\frac{1}{2}(AB)(EC)}{\frac{1}{2}(BD)(DC)} = \frac{(4x)(4y)}{(x)(y)} = 16$$

$$\text{Hence, Area of Triangle AEC} = 16 \times \text{Area of Triangle BDC} = 16 \times 5 = 80$$

$$\text{So, Area of Triangle ABE} = 80 - 5 - 15 = 60$$

Example 2

LMN in a right-angled triangle. Angle NLM = 90°. PQ is parallel to LM. Area of triangle PNQ is 8 cm². Area of triangle LPQ is 16 cm². Work out area of triangle LQM



Recall the following before beginning:

$$\text{Area of triangle} = \frac{1}{2} \times \text{base} \times \text{height}$$

From the area formula we see that

$$\text{Area of PNQ} = \frac{1}{2} \times PQ \times PN = 8,$$

$$\text{Area of LPQ} = \frac{1}{2} \times PQ \times PL = 16$$

Divide these equations and simplify:

$$\frac{\text{Area of PNQ}}{\text{Area of LPQ}} = \frac{\frac{1}{2} \times PQ \times PN}{\frac{1}{2} \times PQ \times PL} = \frac{PN}{PL} = \frac{1}{2} \Rightarrow PL = 2PN$$

From the diagram, notice that:

$$LN = LP + PN = 2PN + PN = 3PN$$

$$\frac{LN}{PN} = 3 = \text{Length Scale Ratio}$$

Hence, Area Scale Factor = (Length Scale Ratio)² = (3)² = 9

We know the Area of PNQ and want the area of NLM. NLM is bigger than PNQ, hence we multiply PNQ by 9.

$$\text{Area of NLM} = 8 \times 9 = 72$$

Finally, note that:

$$\text{Area of NLM} = \text{Area of NPQ} + \text{Area of PQL} + \text{Area of LQM}$$

$$\text{Area of LQM} = \text{Area of NLM} - \text{Area of NPQ} - \text{Area of PQL} = 72 - 8 - 16 = 48 \text{ cm}^2$$

Example 3

Triangle PQR is an enlargement, centre O, of triangle ABC. OAP and OBQ are straight lines. OA = 2 cm, AP = 6 cm, BQ = 7.2 cm, AC = 3.7 cm

i. Work out the length of OB

ii. Work out the length of PR

The area of triangle PQR is 72 cm²

iii. Work out the area of triangle ABC

As given in the question, the two triangles ABC and PQR are similar (as one is the enlargement of the other). Hence, the red, purple, and yellow angles must be equal. As, if two angles of a triangle are equal to two angles of another triangle, then the third angle automatically must be the same. This means B corresponds to Q, C corresponds to R and A corresponds to P.

i. As O is the centre of both triangles, OB must correspond to OQ.

Way 1: Without algebra

We pick a pair of known corresponding sides to find the length scale factor.

$$\text{Length scale factor} = \frac{OP}{OA} = \frac{2+6}{2} = 4$$

OB corresponds to the side OQ, so we can use length OQ to help us find OB. OB is smaller than OQ, so we multiply by the scale factor.

Taking OB as x we solve for x,

$$OB \times 4 = OQ$$

$$x \times 4 = x + 7.2$$

$$x = 2.4$$

$$\text{So, } OB = 2.4 \text{ cm}$$

Way 2: Use algebra

We can write the corresponding lengths as equivalent fractions.

$$\frac{OP}{OA} = \frac{OQ}{OB}$$

$$\frac{2+6}{2} = \frac{OB+7.2}{OB}$$

Now we can cross multiply.

$$2(OB+7.2) = 8OB$$

$$2OB+14.4 = 8OB$$

$$OB = 2.4 \text{ cm}$$

Way 1: Without algebra

Pick a pair of known corresponding sides to find scale factor.

$$\text{Length scale factor} = \frac{OP}{OA} = \frac{2+6}{2} = 4$$

AC corresponds to PR, so we can use length AC to help us find PR. PR is larger than AC, so multiply by the scale factor.

$$PR = 3.7 \times 4 = 14.8$$

$$\text{So, } PR = 14.8 \text{ cm}$$

Way 2: Use algebra

Write corresponding lengths as equivalent fractions.

$$\frac{OP}{OA} = \frac{PR}{AC} \Rightarrow \frac{2+6}{2} = \frac{PR}{3.7}$$

Now we can cross multiply.

$$2PR = 29.6$$

$$\text{So, } PR = 14.8 \text{ cm}$$

iii. Area Scale Factor = (Length Scale Factor)² = (

